

A NOTE RELATING RIDGE REGRESSION AND OLS P-VALUES TO PRECONDITIONED SPARSE PENALIZED REGRESSION

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ABSTRACT. When the design matrix has orthonormal columns, “soft thresholding” the ordinary least squares (OLS) solution produces the Lasso solution [Tibshirani, 1996]. If one uses the Puffer preconditioned Lasso [Jia and Rohe, 2012], then this result generalizes from orthonormal designs to full rank designs (Theorem 1). Theorem 2 refines the Puffer preconditioner to make the Lasso select the same model as removing the elements of the OLS solution with the largest p-values. Using a generalized Puffer preconditioner, Theorem 3 relates ridge regression to the preconditioned Lasso; this result is for the high dimensional setting, $p > n$. Where the standard Lasso is akin to forward selection [Efron et al., 2004], Theorems 1, 2, and 3 suggest that the preconditioned Lasso is more akin to backward elimination. These results hold for sparse penalties beyond ℓ_1 ; for a broad class of sparse and non-convex techniques (e.g. SCAD and MC+), the results hold for all local minima.

1. INTRODUCTION

Preconditioning is a classical computational technique in numerical linear algebra that creates fast algorithms. Several papers have recently proposed and studied the “preconditioned” Lasso [Paul et al., 2008, Huang and Jojic, 2011, Rauhut and Ward, 2011, Jia and Rohe, 2012, Qian and Jia, 2012, Wauthier et al., 2013]. Instead of accelerating standard Lasso algorithms, preconditioning the Lasso creates a new statistical estimator that retains several properties of the Lasso while making the solution less sensitive to the correlation between the columns of the design matrix. This paper demonstrates how penalized least squares estimators with various forms of preconditioning are equivalent to classical quantities in linear regression—the OLS estimator, OLS p-values, and ridge regression.

The theorems below do not make any assumptions on the design matrix beyond full rank. Nor do they assume a linear model $Y = \mathbf{X}\beta + \epsilon$, or assume some conditions on an error term ϵ . Instead of studying the statistical estimation properties of preconditioned penalized least squares problems, the following theorems study the

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preconditioned Lasso estimators as functions of data (\mathbf{X}, Y) that return a vector $\hat{\beta}$. The theorems compare these new functions to classical “functions” like *ols* and ridge regression.

1.1. Preliminaries. While the theorems below do not require the linear model, it is the linear model that motivates the estimators (i.e. functions) studied in this paper. The linear model is

$$(1) \quad Y = \mathbf{X}\beta + \epsilon,$$

where $Y \in R^n$ and $\mathbf{X} \in R^{n \times p}$ are observed, and $\epsilon \in R^n$ is random noise satisfying $E(\epsilon) = 0$ and $E(\epsilon\epsilon') = \sigma^2 I_p$. The goal is to estimate $\beta \in R^p$ with \mathbf{X} and Y .

Throughout the paper, we will assume that \mathbf{X} is full rank (when $n < p$, it is full row rank). Define $\|x\|_q = (\sum_i x_i^q)^{1/q}$. For $n > p$, define

$$\hat{\beta}^{ols} = \arg \min_{b \in R^p} \|Y - \mathbf{X}b\|_2^2 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y \in R^p$$

as the standard OLS estimator.

Definition 1. Define the function $Lasso_\lambda$,

$$Lasso_\lambda(\mathbf{X}, Y) = \arg \min_{b \in R^p} \|Y - \mathbf{X}b\|_2^2 + \lambda \|b\|_1.$$

For $a \in R$, define $\text{sign}(a)$ as $-1, 0$, or 1 , depending on whether a is negative, zero, or positive. Define $(a)^+$ as equal to a if a is nonnegative and equal to zero if a is negative. The soft-thresholding function is defined as

$$t_\lambda(x) = \text{sign}(x)(|x| - \lambda)^+.$$

To apply t_λ to a vector x , apply it element-wise, $[t_\lambda(x)]_j = t_\lambda(x_j)$.

Lemma 1. (Equation 3 in [Tibshirani, 1996]) If the design matrix \mathbf{X} is orthonormal,

$$(2) \quad Lasso_\lambda(\mathbf{X}, Y) = t_\lambda(\hat{\beta}^{ols}).$$

2. PRECONDITIONING THE LASSO

Sections 2 and 3 study the low dimensional setting $n > p$. For these sections, let $\mathbf{X} = UDV'$ be the “skinny” SVD; $U \in R^{n \times p}$ and $V \in R^{p \times p}$ have orthonormal columns and D is a diagonal matrix. The Puffer transform is defined as $F = UD^{-1}U'$ [Jia and Rohe, 2012]. After preconditioning, Equation (1) becomes

$$(3) \quad FY = (F\mathbf{X})\beta + F\epsilon.$$

While $(F\mathbf{X}) = UV'$ is an orthonormal matrix, it is not orthogonalized by rotating the columns as in a QR decomposition; this would correspond to *right* multiplying \mathbf{X} by some matrix. Rotating the columns would create a new basis and make the Lasso penalize in the incorrect basis. By *left* multiplying, each row of $F\mathbf{X}$ is a

linear combination of the rows in \mathbf{X} . Importantly, the regression estimators that use $(F\mathbf{X}, FY)$ instead of (\mathbf{X}, Y) still estimate the same vector β and the Lasso penalizes in the correct basis; the original regression model in Equation (1) contains the exact same β as Equation (3). Define

$$\begin{aligned} Puffer(\mathbf{X}, Y) &= (F\mathbf{X}, FY) \in \mathbb{R}^{n \times p} \times \mathbb{R}^n \quad \text{and} \\ Lasso_\lambda(Puffer(\mathbf{X}, Y)) &= Lasso_\lambda(F\mathbf{X}, FY). \end{aligned}$$

Jia and Rohe [2012] showed that if the smallest singular value of \mathbf{X} is bounded from below, then with p fixed and $n \rightarrow \infty$, the preconditioned Lasso, $Lasso_\lambda(Puffer(\mathbf{X}, Y))$, is sign consistent. Importantly, the Puffer preconditioned Lasso does not require the Irrepresentable Condition from Zhao and Yu [2006]. The next theorem shows how this estimator relates to the classical estimator $\hat{\beta}^{ols} \in \mathbb{R}^p$, computed on the full model. This extends the relationship in Lemma 1 from the case where \mathbf{X} is orthonormal, to the case where \mathbf{X} is full rank.

Theorem 1. *If \mathbf{X} is full rank and $n > p$, then*

$$(4) \quad Lasso_\lambda(Puffer(\mathbf{X}, Y)) = t_\lambda(\hat{\beta}^{ols}).$$

In general, the relationship in Equation (4) does not hold without *Puffer*. All proofs are contained in the Appendix.

3. CORRECTING FOR THE HETEROGENEOUS VARIABILITY IN $\hat{\beta}^{ols}$

Under the linear model,

$$(5) \quad cov(\hat{\beta}^{ols}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}.$$

Define

$$\Sigma^{ols} = cov(\hat{\beta}^{ols}). \quad \text{Then, } variance(\hat{\beta}_j^{ols}) = \Sigma_{jj}^{ols}.$$

If the diagonal elements of Σ^{ols} are not all equal, then some elements of $\hat{\beta}^{ols}$ will have greater variability than others. Classical confidence intervals and p-values account for this uncertainty. However, the preconditioned Lasso estimator above (i.e. $t_\lambda(\hat{\beta}^{ols})$) does not account for this known heteroskedasticity of $\hat{\beta}^{ols}$ by applying a stronger penalty to terms with larger variance.

Classical versions of model selection test the following null hypotheses in various ways:

$$H_{0,j} : \beta_j = 0, \quad \text{for } j \in 1, \dots, p.$$

Under the linear model (1) and $H_{0,j}$,

$$(6) \quad Z_j = \sqrt{n} \frac{\hat{\beta}_j^{ols}}{\sqrt{\sigma^2(\mathbf{X}'\mathbf{X})_{jj}^{-1}}} \xrightarrow{d} N(0, 1) \quad \text{as } n \rightarrow \infty.$$

The classical “marginal” p-values are defined as

$$(7) \quad p_j = 2(1 - \Phi(|Z_j|)), \text{ for } j = 1, \dots, p$$

where Φ is the cdf of the standard normal distribution.

3.1. Scaling matters. Standard Lasso packages normalize the columns of \mathbf{X} so that they all have equal ℓ_2 length. This ensures that each element of the Lasso estimator is equally penalized. However, it does not ensure that the elements of the estimator are equally variable. This section *right* preconditions \mathbf{X} with a diagonal matrix N , making the column lengths heterogeneous. The column lengths are chosen so that the penalty strengths align with the heteroskedasticity of $\hat{\beta}^{ols}$, ensuring equal variability among the elements of the estimator $\hat{\beta}$.

After left and right preconditioning with F and N respectively, the regression equation becomes

$$FY = (F\mathbf{X}N)(N^{-1}\beta) + F\epsilon.$$

Define $\nu \in \mathbb{R}^p$ as the diagonal elements of $(\mathbf{X}'\mathbf{X})^{-1}$ and let $N \in \mathbb{R}^{p \times p}$ be a diagonal matrix with $N_{jj} = \sqrt{\nu_j}$. Because \mathbf{X} is assumed full column rank and N_{jj} is proportional to the standard error of $\hat{\beta}_j^{ols}$, N_{jj} exists and is strictly positive. This ensures that $\text{sign}(N^{-1}\beta) = \text{sign}(\beta)$. So, inferences in the transformed space carry over to the original model.

Take the SVD of $\mathbf{X}N = U_N D_N V_N'$. Then, define $F_N = U_N D_N^{-1} U_N'$ and

$$Puffer_N(\mathbf{X}, Y) = (F_N \mathbf{X} N, F_N Y) \in \mathbb{R}^{n \times p} \times \mathbb{R}^n.$$

Theorem 2 shows that if $\lambda = 1.96\sigma/\sqrt{n}$, then $Lasso_\lambda(Puffer_N(\mathbf{X}, Y))$ selects the same variables as fitting the standard OLS estimator and removing any variables with p-value greater than .05.

Theorem 2. *If \mathbf{X} is full rank and $n > p$, denote*

$$\hat{\beta}^N(\lambda) = Lasso_\lambda(Puffer_N(\mathbf{X}, Y)).$$

Let Z_j be the classical test statistic defined in Equation (6), let p_j be the classical p-value defined in Equation (7), and let Φ represent the cdf of the standard normal distribution.

$$\hat{\beta}_j^N(\lambda) \neq 0 \quad \Leftrightarrow \quad |Z_j| > \lambda\sqrt{n}/\sigma \quad \Leftrightarrow \quad p_j \leq 2(1 - \Phi(\lambda\sqrt{n}/\sigma)).$$

Importantly, this is an algebraic equivalence between $Lasso_\lambda(Puffer_N(\mathbf{X}, Y))$ and the classical OLS p-values in Equation (7). This requires only a full rank design matrix \mathbf{X} and does not make any assumption on the error distribution or the elements of β . Theorem 2 asserts nothing about the statistical reliability of these p-values. If one wishes to use the p-value for statistical inference, then Theorem 2 needs additional assumptions on the error term, ϵ .

3.2. Generalizing to other methods. For simplicity, Theorems 1 and 2 are stated for the Lasso. However, both theorems hold more generally. For $\lambda \geq 0$ and penalty function $\text{pen} : \mathbb{R} \rightarrow \mathbb{R}_+$, define the function $\text{sparse}(\mathbf{X}, Y, \lambda, \text{pen})$ as

$$(8) \quad \text{sparse}(\mathbf{X}, Y, \lambda, \text{pen}) = \arg \min_b \frac{1}{2} \|Y - \mathbf{X}b\|_2^2 + \lambda \sum_j \text{pen}(b_j).$$

Whenever pen has a thresholding function \tilde{t}_λ that satisfies a version of Lemma 1 with orthonormal designs, i.e.

$$\text{sparse}(\mathbf{X}, Y, \lambda, \text{pen}) = \tilde{t}_\lambda(\hat{\beta}^{ols}) \quad \text{for orthonormal } \mathbf{X},$$

then modified versions of Theorems 1 and 2 also apply for this penalty:

$$\begin{aligned} \text{sparse}(\text{Puffer}(\mathbf{X}, Y), \lambda, \text{pen}) &= \tilde{t}_\lambda(\hat{\beta}^{ols}) \\ [\text{sparse}(\text{Puffer}_N(\mathbf{X}, Y), \lambda, \text{pen})]_j &= \tilde{t}_\lambda(\sigma Z_j / \sqrt{n}). \end{aligned}$$

For example, pen could be an ℓ_q penalty for $q \in [0, 1]$, the elastic net penalty, or a concave penalty such as SCAD or MC+ [Fan and Li, 2001, Zhang, 2010]. After preconditioning, a broad class of penalties select the same sequence of models as the Lasso.

4. RELATING TO RIDGE REGRESSION

For $p > n$, take the “skinny” SVD of $\mathbf{X} = UDV'$, where $U \in R^{n \times n}$, $V \in R^{p \times n}$, $D \in R^{n \times n}$. Consider a generalized Puffer transformation

$$F_\tau = U(D^2 + \tau I)^{-1/2}U' \quad \text{and} \quad \text{Puffer}_\tau(\mathbf{X}, Y) = (F_\tau \mathbf{X}, F_\tau Y).$$

Theorem 3 relates $\text{sparse}(\text{Puffer}_\tau(\mathbf{X}, Y), \lambda, \text{pen})$ to the ridge estimator [Hoerl and Kennard, 1970]. For $\tau > 0$, the ridge estimator is

$$(9) \quad \begin{aligned} \hat{\beta}_{\text{ridge}}(\tau) &= \arg \min_b \|Y - \mathbf{X}b\|_2^2 + \tau \|b\|_2^2 \\ &= (\mathbf{X}'\mathbf{X} + \tau I)^{-1} \mathbf{X}'Y. \end{aligned}$$

Define $\hat{\beta}_{\text{ridge}}(0)$ as the Moore-Penrose estimator

$$\begin{aligned} \hat{\beta}_{\text{ridge}}(0) &= \arg \min_{b: \mathbf{X}b=Y} \|b\|_2^2 \\ &= (\mathbf{X}'\mathbf{X})^+ \mathbf{X}'Y, \end{aligned}$$

where $(\mathbf{X}'\mathbf{X})^+$ is the Moore-Penrose pseudoinverse, $VD^{-2}V'$.¹

Define $\mathcal{P}_\tau : \mathbb{R}^p \rightarrow \mathbb{R}^p$ as

$$(10) \quad \mathcal{P}_\tau(v) = \mathbf{X}'(\mathbf{X}\mathbf{X}' + \tau I)^{-1} \mathbf{X}v.$$

¹In what follows, replace any matrix inversion with the Moore-Penrose pseudoinverse if $\tau = 0$ and the true inverse does not exist.

For $\tau = 0$, \mathcal{P}_0 projects onto the row space of \mathbf{X} . Under the linear model (Equation 1), $\hat{\beta}_{ridge}(\tau)$ is an unbiased estimator of $\mathcal{P}_\tau(\beta)$.

The following theorem assumes that *sparse* uses a penalty function satisfying the following assumptions.

Definition 2. Define a function *pen* as a **regular sparse penalty** if it is

- (1) non-differentiable at zero and differentiable everywhere else;
- (2) symmetric, $pen(a) = pen(-a)$;
- (3) monotonically increasing away from zero, $pen(a) \geq pen(b)$ if $|a| > |b|$;
- (4) Lasso derivative in the neighborhood of zero,

$$\lim_{x \rightarrow 0} |pen'(x)| = 1,$$

where pen' is the first derivative of *pen*.

This includes the Lasso, elastic net, SCAD, and MC+. However, ℓ_q penalties fail condition (4) when $q < 1$. Such penalties have an unbounded derivative in the neighborhood of zero which creates discontinuities in the solution path; previous research has also excluded such penalties (e.g. Zhang and Zhang [2012], Loh and Wainwright [2013]).

Theorem 3 says that if *pen* is a regular sparse penalty, then any local minimum for the objective function in $sparse(Puffer_\tau(\mathbf{X}, Y), \lambda, pen)$, transformed by \mathcal{P}_τ , is close to the ridge estimator $\hat{\beta}_{ridge}(\tau)$. Moreover, λ controls the distance between these estimators.

Theorem 3. Let *pen* be a regular sparse penalty (Definition 2) and let $p \geq n$. Let

$$\hat{\beta} = sparse(Puffer_\tau(\mathbf{X}, Y), \lambda, pen).$$

If $\hat{\beta}_j \neq 0$, then

$$\hat{\beta}_{ridge,j}(\tau) - \mathcal{P}_\tau(\hat{\beta})_j = \lambda pen'(\hat{\beta}_j),$$

where pen' is the derivative of *pen*. If $\hat{\beta}_j = 0$, then

$$|\hat{\beta}_{ridge,j}(\tau) - \mathcal{P}_\tau(\hat{\beta})_j| \leq \lambda.$$

Moreover, these results still hold if $\hat{\beta}$ is any local minimizer of the objective function for $sparse(Puffer_\tau(\mathbf{X}, Y), \lambda, pen)$.

Importantly, *sparse* has been computed with the preconditioned data, $Puffer_\tau(\mathbf{X}, Y)$, while $\hat{\beta}_{ridge}(\tau)$ is the traditional estimator computed with the original data. When λ is small, these two estimators are aligned in the row space of \mathbf{X} . Determining the statistically appropriate scale of λ requires some care because after preconditioning,

the scale of the problem changes; $\|\mathbf{X}\|_F^2 = O(np)$, but $\|F_\tau \mathbf{X}\|_F^2 = O(n)$. A forthcoming revision to Jia and Rohe [2012], shows that $\text{sparse}(\text{Puffer}_0(\mathbf{X}, Y), \lambda, \|\cdot\|_1)$ is sign consistent when $\min_j \beta_j$ is larger than $\log n / \sqrt{n}$ and

$$\lambda = O\left(\left(\frac{\log n \log p}{p}\right)^{1/2}\right).$$

In the high dimensional setting, this λ is clearly converging to zero.

If Theorem 3 held without \mathcal{P}_τ , then it would say that $\text{Lasso}_\lambda(\text{Puffer}_\tau(\mathbf{X}, Y)) = t_\lambda(\hat{\beta}_{\text{ridge}}(\tau))$, which would make a clear analogy to backward elimination. However, the inclusion of \mathcal{P}_τ stains the analogy to backward elimination. In fact, it is not even true that $\mathcal{P}_\tau(\text{Lasso}_\lambda(\text{Puffer}_\tau(\mathbf{X}, Y))) = t_\lambda(\hat{\beta}_{\text{ridge}}(\tau))$; the left side is in the row space of \mathbf{X} , but the right side is not. That said, typical algorithms to compute the Lasso solution path (e.g. Efron et al. [2004]) start at $\lambda = \infty$ with $\text{Lasso}_{\lambda=\infty}(\mathbf{X}, Y) = 0$ and decrease λ , increasing the number of terms in the model. This resembles forward selection. The results of Theorem 3 are “backwards” in the sense that for $\lambda \rightarrow 0$, $\mathcal{P}_\tau(\text{Lasso}_\lambda(\text{Puffer}_\tau(\mathbf{X}, Y))) \rightarrow \hat{\beta}_{\text{ridge}}(\tau)$.

Suppose $\hat{\beta}^{(1)}$ and $\hat{\beta}^{(2)}$ are both local minima of $\text{sparse}(\text{Puffer}_0(\mathbf{X}, Y), \lambda, \text{pen})$ for a regular sparse penalty $\text{pen}(x)$ that is concave on $\{x : x > 0\}$. By concavity and the definition of regular sparse penalty, it follows that $|\text{pen}'(x)| \leq 1$ for $x \neq 0$. So, the triangle inequality around $\hat{\beta}_{\text{ridge},j}(0)$ yields

$$(11) \quad \mathcal{P}_0(\hat{\beta}_j^{(1)} - \hat{\beta}_j^{(2)}) \leq 2\lambda.$$

So after preconditioning, local minima are exceedingly similar in the row space of \mathbf{X} . Moreover, even if $\hat{\beta}^{(1)}$ and $\hat{\beta}^{(2)}$ are (local) minima from different penalty functions, then Equation 11 holds so long as (i) both pen functions are regular and concave and (ii) both are computed with the same tuning parameter λ .² In general, β is not identifiable when $p > n$; only the projection of β into the row space of \mathbf{X} is identifiable [Shao and Deng, 2012]. This suggests that, after preconditioning, it is difficult statistically distinguish the difference between local minima or the difference between penalty functions.

5. DISCUSSION

Several previous papers have studied different preconditioners for the Lasso [Paul et al., 2008, Huang and Jovic, 2011, Rauhut and Ward, 2011, Jia and Rohe, 2012, Qian and Jia, 2012, Wauthier et al., 2013]. This paper connects two types of preconditioning techniques to the classical OLS solution. When performing model selection in the classical setting of $n \gg p$, the marginal p-values from OLS are typically considered

²If they have different tuning parameters λ_1 and λ_2 , then the bound 2λ is replaced by $\lambda_1 + \lambda_2$.

more informative than the absolute sizes of the elements in $\hat{\beta}^{ols}$; the p-values account for the potentially heterogeneous standard errors across $\hat{\beta}^{ols}$. This suggests that $Lasso_{\lambda}(Puffer_N(\mathbf{X}, Y))$ should be preferred to $Lasso_{\lambda}(Puffer(\mathbf{X}, Y))$. However, classical intuitions also suggest that two variables might have statistically insignificant p-values because these variables are correlated with each other (see Section 10.1 in Weisberg [2014]). As such, a backward procedure should have multiple steps. On each step, remove the variable with the largest p-value and refit the OLS with the remaining predictors. Neither of the preconditioning techniques above creates a Lasso solution path that is equivalent to this multi-step approach. A preconditioner that is designed to match this path must be a function of Y and λ , thus becoming much more complicated.

The Lasso (without preconditioning) is akin to forward selection [Efron et al., 2004]. Classical methods of forward selection are not model selection consistent, unless the columns of the design matrix are only weakly correlated [Tropp and Gilbert, 2007]. Similarly, the Lasso is not sign consistent unless the design matrix satisfies the Irrepresentable Condition [Zhao and Yu, 2006]. After preconditioning with the (generalized) Puffer transformation, Theorems 1, 2, and 3 show that the Lasso is akin to backward elimination. This preconditioner makes the design orthogonal when $n > p$, trivially satisfying all consistency conditions (e.g. Irrepresentable Condition, RIP, etc.). As such, Theorem 1 implies that a one step backward elimination is also sign consistent.

This suggests that backward procedures are sign consistent when forward procedures are not. However, backward procedures are not a panacea. They will become unstable whenever the OLS p-values are incorrect (e.g. when p is large compared to n and the CLT does not hold for the test statistic Z_j in Equation (6)).

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APPENDIX A.

Proof for Theorem 1

Proof. Define the function ols ,

$$ols(\tilde{\mathbf{X}}, \tilde{Y}) = (\tilde{\mathbf{X}}' \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}' \tilde{Y}.$$

$F\mathbf{X}$ has orthonormal columns. So, by Lemma 1,

$$Lasso_{\lambda}(F\mathbf{X}, FY) = t_{\lambda}(ols(F\mathbf{X}, FY)).$$

Again using the fact that $F\mathbf{X}$ has orthonormal columns, $ols(F\mathbf{X}, FY) = \hat{\beta}^{ols}$;

$$\begin{aligned} ols(F\mathbf{X}, FY) &= ((F\mathbf{X})'F\mathbf{X})^{-1}(F\mathbf{X})'FY \\ &= (F\mathbf{X})'FY \\ &= VD^{-1}U'Y \\ &= (VD^2V')^{-1}VDU'Y = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y = \hat{\beta}^{ols}. \end{aligned}$$

□

Proof for Theorem 2

Proof. Define $\mathbf{X}_N = \mathbf{X}N$; its Puffer transformation is F_N . Using Theorem 1,

$$\begin{aligned} \hat{\beta}^N(\lambda) = Lasso_\lambda(F_N\mathbf{X}_N, F_NY) &= Lasso_\lambda(F_N\mathbf{X}_N, F_NY) \\ &= t_\lambda(ols(\mathbf{X}_N, Y)) = t_\lambda(ols(\mathbf{X}N, Y)). \end{aligned}$$

Then,

$$ols(\mathbf{X}N, Y) = ((\mathbf{X}N)'(\mathbf{X}N))^{-1}(\mathbf{X}N)'Y = N^{-1}(\mathbf{X}'\mathbf{X})^{-1}N^{-1}NX'Y = N^{-1}\hat{\beta}^{ols}$$

and the j th element of this is

$$[ols(\mathbf{X}N, Y)]_j = \frac{\hat{\beta}_j^{ols}}{\sqrt{(\mathbf{X}'\mathbf{X})_{jj}^{-1}}} = \sigma Z_j / \sqrt{n},$$

where Z_j is the test statistic for $H_{0,j}$ defined in Equation (6).

Putting this together, $[\hat{\beta}^N(\lambda)]_j = t_\lambda(\sigma Z_j / \sqrt{n})$. So,

$$\hat{\beta}_j^N(\lambda) \neq 0 \Leftrightarrow |Z_j| > \lambda\sqrt{n}/\sigma \Leftrightarrow p_j \leq 2(1 - \Phi(\lambda\sqrt{n}/\sigma)).$$

□

The proof for Theorem 3 relies on the following lemma.

Lemma 2. *If $p \geq n$, then for any vector $v \in \mathbb{R}^p$,*

$$\begin{aligned} \mathcal{P}_\tau(v) &= (F_\tau\mathbf{X})'F_\tau\mathbf{X}v \\ \hat{\beta}_{ridge}(\tau) &= (F_\tau\mathbf{X})'F_\tau Y, \end{aligned}$$

where \mathcal{P}_τ is defined in Equation (10), and $\hat{\beta}_{ridge}(\tau)$ is defined in Equation (9).

A proof of Lemma 2 follows this proof of Theorem 3.

Proof. If $\hat{\beta}$ is a local minimizer of

$$\frac{1}{2}\|F_\tau Y - F_\tau\mathbf{X}b\|_2^2 + \lambda \sum_j \text{pen}(b_j),$$

then,

$$(12) \quad (F_\tau \mathbf{X})' F_\tau Y - (F_\tau \mathbf{X})' F_\tau \mathbf{X} \hat{\beta} - \lambda \partial \text{pen}(\hat{\beta}) = 0,$$

where $\text{pen}(\hat{\beta}) \in \mathbb{R}^p$ is defined as $[\text{pen}(\hat{\beta})]_j = \text{pen}(\hat{\beta}_j)$ and $\partial \text{pen}(\hat{\beta})$ is a generalized subgradient of $\text{pen}(\hat{\beta})$ [Clarke, 1990]. By the assumption that pen is a regular sparse penalty, if $x \neq 0$, then $\partial \text{pen}(x) = \text{pen}'(x)$ and $\partial \text{pen}(0) \in [-1, 1]$.

Substituting the results from Lemma 2 into Equation 12 gives the result. \square

The following proves Lemma 2.

Proof. When $p \geq n$, the matrix $U \in \mathbb{R}^{n \times n}$ is orthonormal. So,

$$F_\tau' F_\tau = U(D^2 + \tau I)^{-1} U' = (U D^2 U' + \tau U U')^{-1} = (\mathbf{X} \mathbf{X}' + \tau I)^{-1}.$$

Using this,

$$\begin{aligned} \mathcal{P}_\tau(v) &= \mathbf{X}' (\mathbf{X} \mathbf{X}' + \tau I)^{-1} \mathbf{X} v \\ &= \mathbf{X}' F_\tau' F_\tau \mathbf{X} v \\ &= (F_\tau \mathbf{X})' F_\tau \mathbf{X} v. \end{aligned}$$

Let $\mathbf{X} = U \tilde{D} \tilde{V}'$ be the “full” SVD with $\tilde{V} \in \mathbb{R}^{p \times p}$ and $\tilde{D} \in \mathbb{R}^{n \times p}$. Notice that U is unchanged. In the following calculations, the identity matrix I takes a subscript denoting its dimension, $I_d \in \mathbb{R}^{d \times d}$.

$$\begin{aligned} (F_\tau \mathbf{X})' F_\tau Y &= V D U' U (D^2 + \tau I_n)^{-1} U' Y \\ &= V (D^2 + \tau I_n)^{-1} D U' Y \\ &= \tilde{V} (\tilde{D}' \tilde{D} + \tau I_p)^{-1} \tilde{D}' U' Y \\ &= \tilde{V} (\tilde{D}' \tilde{D} + \tau I_p)^{-1} \tilde{V}' \tilde{V} \tilde{D}' U' Y \\ &= (\tilde{V} \tilde{D}' \tilde{D} \tilde{V}' + \tau \tilde{V} \tilde{V}')^{-1} \tilde{V} \tilde{D}' U' Y \\ &= (\mathbf{X}' \mathbf{X} + \tau I_p)^{-1} \mathbf{X}' Y \\ &= \hat{\beta}_{\text{ridge}}(\tau). \end{aligned}$$

\square